Simulation of 2D Driven Cavity Flow with Compact Scheme

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Abstract

This paper proposes a higher order scheme in a compact 3×3 stencil in fusion with one sided formula for solving 2D unsteady Lid driven cavity flow at moderate Reynolds Number. This problem is a typical example for incompressible viscous flows in a constricted domain. The developed implicit scheme is temporally first order, spatially fourth order and requires a transformation from physical to computational domain. The scheme is capable of establishing results supporting the characteristics of the counter rotating vortices and bounded shear layers on the wall. The numerical data is compared with the established data of elsewhere studies. Detailed comparison data obtained by the scheme is presented.

Keywords: *N-S equations, Compact scheme, Driven cavity flow*

1.0 Introduction

Partial differential equations which are elliptic in nature plays a major role in the field of incompressible fluid flow problems. The governing equations in the stream function - vorticity formulation are more convenient and are solved using higher order compact schemes, spectral collocation methods, pseudo spectral methods etc.

High order accuracy with compact grids attracted many researchers and a few well known finite difference schemes are contributed by S Abarbanel and A Kumar [1], T Y Hou and B R Wetton [2], and Z F Tian [3]. Compact schemes based on nine point stencil are developed by U Ghia et al. [4], J C Kalita et al. [5], W F Spotz and G F Carey [6]. M Li et al. [7] introduced a scheme called genuine compactness where the stream function equation is used to get a compact scheme for vorticity equation in 3 × 3 molecule. In the recent years compact schemes for Navier-Stokes equation in body fitted coordinates are popular. YVSS Sanyasiraju and V Manjula [8] proposed a higher order semi compact scheme which is an extended idea of E Weinen and J G Liu [9]. Compact schemes in curvilinear systems reduces the algebraic complexity but also serves the goal of accuracy and stability. In the present study an attempt is made to

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use the nice features of one sided differences in the derivation of higher compact scheme for 2D driven cavity flow problem. The compactness of every term involving stream function and vorticity is reserved and few terms involving nodes adjacent to the boundary of computational domain are calculated using one sided differences. The advantage of this scheme is that the extra nodal points are completely avoided in the calculation of vorticity.

2.0 Basic Governing Equations

Consider the flow of an unsteady isothermal incompressible fluid in the region $\mathrm{R} = \Omega \times (0,\mathrm{T}), \mathrm{T} > 0,\ \Omega \subset R^2$ with Γ as its boundary. In this region R the flow is governed by the equations (1) – (2).

Continuity equation:

$$\nabla . \boldsymbol{u} = 0 \tag{1}$$

Momentum equation:

$$\boldsymbol{u}_t + \nabla p + \boldsymbol{u}.\nabla \boldsymbol{u} = \frac{1}{Re}\nabla^2 \boldsymbol{u} \tag{2}$$

Here $\boldsymbol{u} = (u, v)$ where u *and* v are the velocities along x and y axis respectively, p is the pressure and the non-dimensionless parameter is the Reynolds number $Re = \frac{UL}{\nu}$ where U is the characteristic velocity, L is the characteristic length and ν is the kinematic viscosity. The stream function - vorticity formulation of the governing equations in the absence of pressure are:

$$\nabla^2_{(x,y)}\psi = -\omega \tag{3}$$

$$\frac{\partial \omega}{\partial t} + \left(u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right) = \frac{1}{Re}\nabla^2_{(x,y)}\omega \tag{4}$$

ψ represents stream function, $\omega = u_y - v_x$ represents vorticity. The relation between the velocities and stream function is expressed as

$$u = \frac{\partial \psi}{\partial y}\ ,\ v = -\frac{\partial \psi}{\partial x} \tag{5}$$

Equation (3) is a Poisson equation for stream function and equation (4) is a parabolic-type vorticity transport equation. To obtain appropriate body fitted coordinates for the governing equations the Cartesian system (x, y) is converted to curvilinear system (ζ, η) in a manner that they satisfy the Laplace equations $\nabla^2\zeta = 0, \nabla^2\eta = 0$.

The governing equations in the transformed body fitted coordinates (ζ, η) plane is given by equations (6) to (11).

$$l\frac{\partial^2\psi}{\partial\zeta^2}+m\frac{\partial^2\psi}{\partial\zeta\partial\eta}+n\frac{\partial^2\psi}{\partial\eta^2} = -\omega \tag{6}$$

$$\frac{\partial\omega}{\partial t}-\frac{1}{Re}\left(l\frac{\partial^2\omega}{\partial\zeta^2}+m\frac{\partial^2\omega}{\partial\zeta\partial\eta}+n\frac{\partial^2\omega}{\partial\eta^2}\right)+p\frac{\partial\omega}{\partial\zeta}+q\frac{\partial\omega}{\partial\eta} = 0 \tag{7}$$

where the contravariant components of velocity in the transformed plane are as follows:

$$u=\frac{1}{J}\left(x_\zeta\psi_\eta-x_\eta\psi_\zeta\right),\;\; v=-\frac{1}{J}\left(y_\eta\psi_\zeta-y_\zeta\psi_\eta\right) \tag{8}$$

$$l=\frac{1}{J^2}\left(x_\eta^2+y_\eta^2\right), m= -\frac{2}{J^2}\left(y_\zeta y_\eta+x_\zeta x_\eta\right),\; n= -\frac{1}{J^2}\left(x_\zeta^2+ y_\zeta^2\right) \tag{9}$$

with $\quad p=uy_\eta-vx_\eta+r,\;\; q= -uy_\zeta+vx_\zeta+s,\; J=x_\zeta y_\eta-x_\eta y_\zeta$

$$r= \frac{1}{J^3}\left\{-y_\eta\left(\left(x_\eta^2+ y_\eta^2\right)x_{\zeta\zeta}-2\left(y_\zeta y_\eta+x_\zeta x_\eta\right)x_{\zeta\eta}+\left(x_\zeta^2+ y_\zeta^2\right)x_{\eta\eta}\right)\right\}$$
$$+$$
$$\frac{1}{J^3}\left\{x_\eta\left(\left(x_\eta^2+ y_\eta^2\right)y_{\zeta\zeta}-2\left(y_\zeta y_\eta+x_\zeta x_\eta\right)y_{\zeta\eta}+\left(x_\zeta^2+ y_\zeta^2\right)y_{\eta\eta}\right)\right\} \tag{10}$$

$$s= \frac{1}{J^3}\left\{-y_\zeta\left(\left(x_\eta^2+ y_\eta^2\right)x_{\zeta\zeta}-2\left(y_\zeta y_\eta+x_\zeta x_\eta\right)x_{\zeta\eta}+\left(x_\zeta^2+ y_\zeta^2\right)x_{\eta\eta}\right)\right\}+$$
$$\frac{1}{J^3}\left\{x_\zeta\left(\left(x_\eta^2+ y_\eta^2\right)y_{\zeta\zeta}-2\left(y_\zeta y_\eta+x_\zeta x_\eta\right)y_{\zeta\eta}+\left(x_\zeta^2+ y_\zeta^2\right)y_{\eta\eta}\right)\right\} \tag{11}$$

3.0 Discretisation of Governing equations

The higher order compact scheme for stream function and vorticity is obtained using the operator defined for Poisson equations J C Strikwerda [10]. The discretized equations are

$$l\,D_\zeta^2\left(1+\frac{h^2}{12}D_\eta^2\right)\psi+mD_\zeta D_\eta\left(1-\frac{h^2}{12}\left(D_\zeta^2+D_\eta^2\right)\right)\psi+nD_\eta^2\left(1+\right.$$
$$\left.\frac{h^2}{12}D_\zeta^2\right)\psi=-\left(1+\frac{h^2}{12}\left(D_\zeta^2+D_\eta^2\right)\right)\omega+O(h^4) \tag{12}$$

Let $\left(1+\frac{h^2}{12}\left(D_\zeta^2+D_\eta^2\right)\right)\omega = \widetilde{\omega}$

$$\frac{\partial\tilde{\omega}}{\partial t} = \frac{1}{Re}\left(l\, D_\zeta^2\left(1+\frac{h^2}{12}D_\eta^2\right)\omega \;+ mD_\zeta D_\eta\left(1-\frac{h^2}{12}\left(D_\zeta^2+D_\eta^2\right)\right)\omega \right.$$
$$\left. + nD_\eta^2\left(1+\frac{h^2}{12}D_\zeta^2\right)\omega\right)$$
$$-\frac{1}{J}\left(1+\frac{h^2}{12}\left(D_\zeta^2+D_\eta^2\right)\right)\left(\frac{\partial\psi}{\partial\eta}\frac{\partial\omega}{\partial\zeta}-\frac{\partial\psi}{\partial\zeta}\frac{\partial\omega}{\partial\eta}\right)+O(h^4) \qquad (13)$$

The system of equations (12) and (13) leads to a compact scheme since the metric coefficient "*m*" vanishes with the choice of orthogonal transformation. The main focus is to discretise the last term in equation (13) which needs extra nodes as it involves third order derivatives. The third order derivatives of stream function and vorticity are expressed using one sided formula only at the nodes adjacent to the four boundaries of the computational domain, consequently the compactness is preserved. The one sided difference formula for third order derivatives are derived using Lagrange's interpolation formula A K Singh and G R Thorpe [11]. The system of equations is solved using outer inner algorithm and point SOR method.

4.0 Driven Cavity Problem

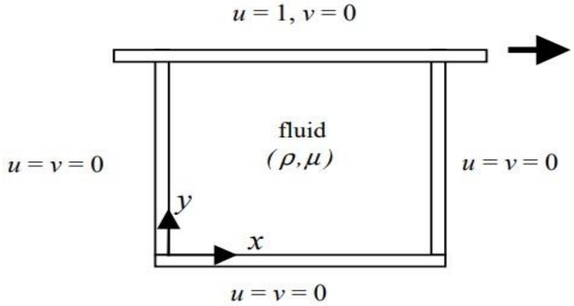

Fig. 1. Physical configuration

The two dimensional lid-driven cavity is a benchmark problem used to validate numerical codes of the 2D Navier-Stokes equations (Fig. 1). Consider a cavity within a square (0,1) × (0,1). Due to the simple geometry the computational domain is under the region (0,1) × (0,1). There are three stationary walls and the top wall moves from left to right and induce the fluid to move. Hence the velocities on the stationary walls are $u = v = 0$, and the velocity on the top wall is $u = 1, v = 0$. The boundary conditions for vorticity is evaluated using Jenson's Formula [8]. Numerical computations are performed using uniform grids of size

81 × 81 and 101 × 101 with time step of 0.001 for Re = 1000. Fig. 2 depicts the stream line evolution of Re = 1000 at different time levels. Fig. 2 shows clearly that the scheme can capture the shear layers of vortex contours in the left and right bottom of the cavity. Fig. 3 describes the vorticity values on the top wall at Re = 1000. The present result shows a very good comparison with the benchmark result of [4]. The steady state vorticity on the walls other than the top boundary is displayed in Fig. 4.

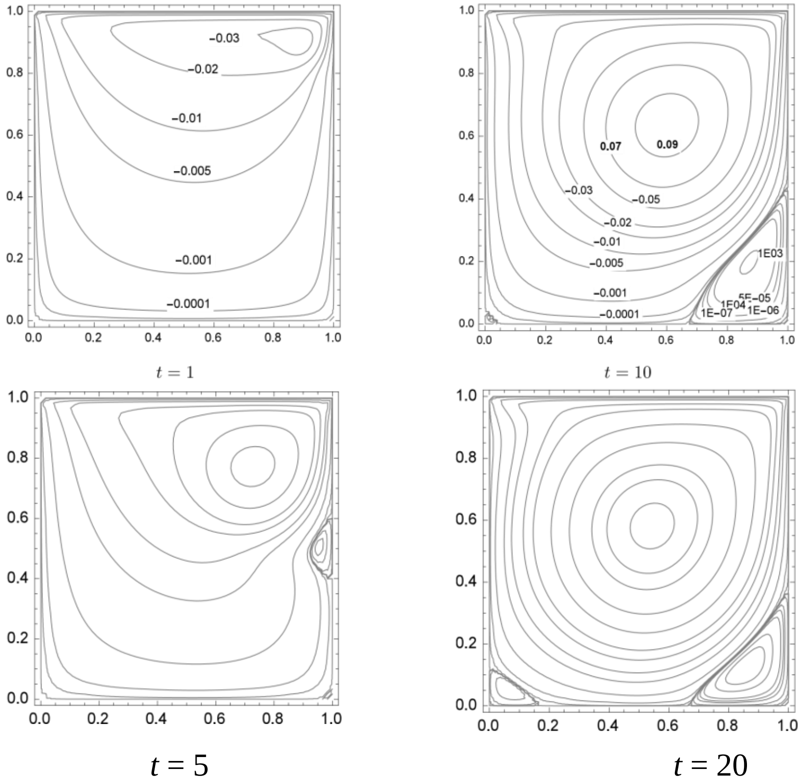

Fig. 2. Evolution of streamlines at different time levels for Re = 1000

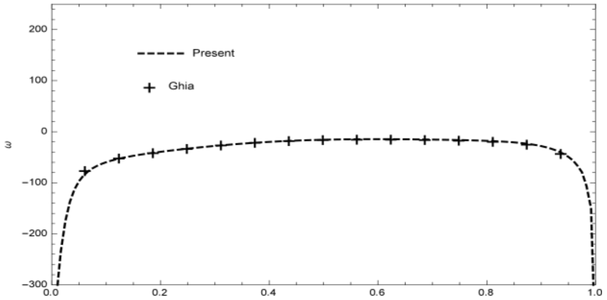

Fig. 3. Vorticity values on the moving wall boundary at Re = 1000

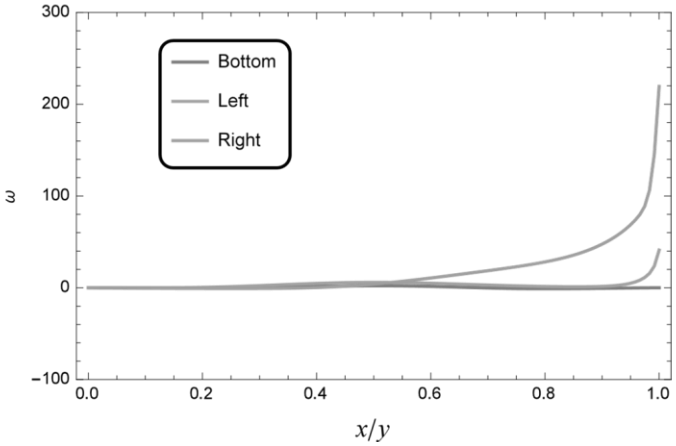

Fig. 4. Steady-state vorticity on the Bottom, Left, Right walls at *Re* = 1000

5.0 Conclusion

In this study a fourth order compact scheme with one sided difference for a few terms was introduced in the body fitted coordinate system of incompressible Navier-Stokes equations. The results of test case exhibited reliability of the scheme to obtain the solution. The major advantage is that the computation of ghost points are completely avoided. All the simulations were performed with fewer grids of size (101 × 101). This evidences that the scheme is efficient for generating appropriate solutions.

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